HOMOGENIZATION OF THE ELLIPTIC DIRICHLET PROBLEM: OPERATOR ERROR ESTIMATES IN L_2

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ABSTRACT. Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain of class C^2. In the Hilbert space $L_2(\mathcal{O};\mathbb{C}^n)$, we consider a matrix elliptic second order differential operator $\mathcal{A}_{D,\varepsilon}$ with the Dirichlet boundary condition. Here $\varepsilon > 0$ is the small parameter. The coefficients of the operator are periodic and depend on $\mathbf{x}/\varepsilon$. A sharp order operator error estimate $\|\mathcal{A}_{D,\varepsilon}^{-1} - (\mathcal{A}_D^0)^{-1}\|_{L_2 \to L_2} \le C\varepsilon$ is obtained. Here $\mathcal{A}_D^0$ is the effective operator with constant coefficients and with the Dirichlet boundary condition.

INTRODUCTION

The paper concerns homogenization theory of periodic differential operators (DO's). A broad literature is devoted to homogenization problems in the small period limit. First of all, we mention the books [BeLPa], [BaPan], [ZhKO].

0.1. Operator-theoretic approach to homogenization problems. In a series of papers [BSu1-3] by M. Sh. Birman and T. A. Suslina a new operator-theoretic (spectral) approach to homogenization problems was suggested and developed. By this approach, the so-called operator error estimates in homogenization problems for elliptic DO's were obtained. Matrix elliptic DO's acting in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ and admitting a factorization of the form $\mathcal{A}_\varepsilon = b(\mathbf{D})^* g(\mathbf{x}/\varepsilon) b(\mathbf{D})$, $\varepsilon > 0$, were studied. Here $g(\mathbf{x})$ is a periodic matrix-valued function and $b(\mathbf{D})$ is a first order DO. The precise assumptions on $g(\mathbf{x})$ and $b(\mathbf{D})$ are described below in Section 1.

In [BSu1-3], the equation $\mathcal{A}_\varepsilon \mathbf{u}_\varepsilon + \mathbf{u}_\varepsilon = \mathbf{F}$, where $\mathbf{F} \in L_2(\mathbb{R}^d;\mathbb{C}^n)$, was considered. The behavior of the solution $\mathbf{u}_\varepsilon$ for small ε was studied. The solution $\mathbf{u}_\varepsilon$ converges in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to the solution $\mathbf{u}_0$ of the "homogenized" equation $\mathcal{A}^0 \mathbf{u}_0 + \mathbf{u}_0 = \mathbf{F}$, as $\varepsilon \to 0$. Here $\mathcal{A}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$ is the *effective operator* with the constant effective matrix g^0. In [BSu1], it was proved that

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathbb{R}^d)} \le C\varepsilon \|\mathbf{F}\|_{L_2(\mathbb{R}^d)}.$$

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In operator terms it means that the resolvent $(\mathcal{A}_\varepsilon + I)^{-1}$ converges in the operator norm in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to the resolvent of the effective operator, as $\varepsilon \to 0$, and

$$\|(\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A}^0 + I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le C\varepsilon. \tag{0.1}$$

In [BSu2], more accurate approximation of the resolvent $(\mathcal{A}_\varepsilon + I)^{-1}$ in the operator norm in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ with an error term $O(\varepsilon^2)$ was obtained.

In [BSu3], approximation of the resolvent $(\mathcal{A}_\varepsilon + I)^{-1}$ in the norm of operators acting from $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to the Sobolev space $H^1(\mathbb{R}^d;\mathbb{C}^n)$ was found:

$$\|(\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A}^0 + I)^{-1} - \varepsilon K(\varepsilon)\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \le C\varepsilon; \tag{0.2}$$

this corresponds to approximation of $\mathbf{u}_\varepsilon$ in the "energy" norm. Here $K(\varepsilon)$ is a corrector. It contains rapidly oscillating factors and so depends on ε.

Estimates (0.1), (0.2) are called the *operator error estimates*. They are order-sharp; the constants in estimates are controlled explicitly in terms of the problem data. The method of [BSu1–3] is based on the scaling transformation, the Floquet-Bloch theory and the analytic perturbation theory.

0.2. A different approach to operator error estimates in homogenization problems was suggested by V. V. Zhikov. In [Zh1, Zh2, ZhPas, Pas], the scalar elliptic operator $-\operatorname{div} g(\mathbf{x}/\varepsilon)\nabla$ (where $g(\mathbf{x})$ is a matrix with real entries) and the system of elasticity theory were studied. Estimates of the form (0.1), (0.2) for the corresponding problems in $\mathbb{R}^d$ were obtained. The method was based on analysis of the first order approximation to the solution and introducing of an additional parameter. Besides the problems in $\mathbb{R}^d$, homogenization problems in a bounded domain $\mathcal{O} \subset \mathbb{R}^d$ with the Dirichlet or Neumann boundary condition were studied. Approximation of the solution in $H^1(\mathcal{O})$ was deduced from the corresponding result in $\mathbb{R}^d$. Due to the "boundary layer" influence, estimates in a bounded domain become worse and the error term is $O(\varepsilon^{1/2})$. The estimate $\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathcal{O})} \le C\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O})}$ follows from approximation of the solution in $H^1(\mathcal{O})$ by roughening.

Similar results for the operator $-\operatorname{div} g(\mathbf{x}/\varepsilon)\nabla$ in a bounded domain with the Dirichlet or Neumann boundary condition were obtained in the papers [Gr1, Gr2] by G. Griso by the "unfolding" method.

0.3. Approximation of the resolvent in the $(L_2 \to H^1)$-norm. The present paper relies on the results of [PSu]. In that paper, matrix DO's $\mathcal{A}_{D,\varepsilon}$ in a bounded domain $\mathcal{O} \subset \mathbb{R}^d$ of class C^2 were studied. The operator $\mathcal{A}_{D,\varepsilon}$ is defined by the differential expression $b(\mathbf{D})^* g(\mathbf{x}/\varepsilon) b(\mathbf{D})$ with the Dirichlet condition on $\partial\mathcal{O}$. The effective operator $\mathcal{A}_D^0$ is given by the expression $b(\mathbf{D})^* g^0 b(\mathbf{D})$ with the Dirichlet boundary condition. The behavior for small ε of the solution $\mathbf{u}_\varepsilon$ of the equation $\mathcal{A}_{D,\varepsilon}\mathbf{u}_\varepsilon = \mathbf{F}$, where $\mathbf{F} \in L_2(\mathcal{O};\mathbb{C}^n)$, is studied. Estimates for the H^1-norm of the difference of the solution $\mathbf{u}_\varepsilon$ and its first order approximation are obtained. By roughening of this result, an estimate for $\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathcal{O})}$ is proved. Here $\mathbf{u}_0$ is the solution of the equation $\mathcal{A}_D^0\mathbf{u}_0 = \mathbf{F}$.

In operator terms, the following estimates are obtained:

$$\|\mathcal{A}_{D,\varepsilon}^{-1} - (\mathcal{A}_D^0)^{-1} - \varepsilon K_D(\varepsilon)\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})} \le C\varepsilon^{1/2}, \tag{0.3}$$

$$\|\mathcal{A}_{D,\varepsilon}^{-1} - (\mathcal{A}_D^0)^{-1}\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \le C\varepsilon^{1/2}. \tag{0.4}$$

Here $K_D(\varepsilon)$ is the corresponding corrector.

The method of [PSu] is based on using estimates (0.1), (0.2) for homogenization problem in $\mathbb{R}^d$ obtained in [BSu1,3] and on the tricks suggested in [Zh2], [ZhPas] that allow one to deduce estimate (0.3) from (0.1), (0.2). Main difficulties are related to estimating of the "discrepancy" $\mathbf{w}_\varepsilon$, which satisfies the equation $\mathcal{A}_\varepsilon \mathbf{w}_\varepsilon = 0$ in $\mathcal{O}$ and the boundary condition $\mathbf{w}_\varepsilon = \varepsilon K_D(\varepsilon)\mathbf{F}$ on $\partial\mathcal{O}$.

0.4. The main result. It must be mentioned that estimate (0.4) is quite a rough consequence of (0.3). So, the refinement of estimate (0.4) is a natural problem. In [ZhPas], for the case of the scalar elliptic operator $-\operatorname{div} g(\mathbf{x}/\varepsilon)\nabla$ (where $g(\mathbf{x})$ is a matrix with real entries) an estimate for $\|\mathcal{A}_{D,\varepsilon}^{-1} - (\mathcal{A}_D^0)^{-1}\|_{L_2\to L_2}$ of order $\varepsilon^{\frac{d}{2d-2}}$ for $d \ge 3$ and of order $\varepsilon|\log\varepsilon|$ for $d = 2$ was obtained. The proof essentially relies on using the maximum principle which is specific for scalar elliptic equations.

In the present paper, we prove a *sharp order operator error estimate*

$$\|\mathcal{A}_{D,\varepsilon}^{-1} - (\mathcal{A}_D^0)^{-1}\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \le C\varepsilon. \tag{0.5}$$

Estimate (0.5) for matrix elliptic DO's refines even the known classical (non-operator) error estimates.

Method of the proof relies on the results and technique of [PSu]. The problem reduces to estimating of the L_2-norm of $\mathbf{w}_\varepsilon$. Using of the operator approach and duality arguments is important. Employing approximation of the resolvent $\mathcal{A}_{D,\varepsilon}^{-1}$ in the norm of operators acting from $L_2(\mathcal{O};\mathbb{C}^n)$ to $H_0^1(\mathcal{O};\mathbb{C}^n)$, we find approximation of the same operator in the norm of operators acting from $H^{-1}(\mathcal{O};\mathbb{C}^n)$ to $L_2(\mathcal{O};\mathbb{C}^n)$. The last approximation combined with the boundary layer estimates allows one to obtain the required estimate for the L_2-norm of $\mathbf{w}_\varepsilon$.

0.5. The plan of the paper. The paper contains three sections. In Section 1, the class of operators is introduced, the effective operator is described, and the main result is formulated. Section 2 contains some auxiliary statements needed for further investigation. In Section 3, the main result is proved.

0.6. Notation. Let $\mathfrak{H}$ and $\mathfrak{H}_*$ be complex separable Hilbert spaces. The symbols $(\cdot,\cdot)_{\mathfrak{H}}$ and $\|\cdot\|_{\mathfrak{H}}$ stand for the inner product and the norm in $\mathfrak{H}$; the symbol $\|\cdot\|_{\mathfrak{H}\to\mathfrak{H}_*}$ denotes the norm of a linear continuous operator acting from $\mathfrak{H}$ to $\mathfrak{H}_*$.

The symbols $\langle\cdot,\cdot\rangle$ and $|\cdot|$ stand for the inner product and the norm in $\mathbb{C}^n$; $\mathbf{1} = \mathbf{1}_n$ is the identity $(n\times n)$-matrix. We use the notation $\mathbf{x} = (x_1,\dots,x_d) \in \mathbb{R}^d$, $iD_j = \partial_j = \partial/\partial x_j$, $j = 1,\dots,d$, $\mathbf{D} = -i\nabla = (D_1,\dots,D_d)$. The L_p-classes of $\mathbb{C}^n$-valued functions in a domain $\mathcal{O} \subset \mathbb{R}^d$ are denoted by $L_p(\mathcal{O};\mathbb{C}^n)$,

$1 \le p \le \infty$. The Sobolev classes of $\mathbb{C}^n$-valued functions in a domain $\mathcal{O} \subset \mathbb{R}^d$ are denoted by $H^s(\mathcal{O};\mathbb{C}^n)$. By $H^1_0(\mathcal{O};\mathbb{C}^n)$ we denote the closure of $C_0^\infty(\mathcal{O};\mathbb{C}^n)$ in $H^1(\mathcal{O};\mathbb{C}^n)$. If $n=1$, we write simply $L_p(\mathcal{O})$, $H^s(\mathcal{O})$, etc., but sometimes we use such abbreviated notation also for spaces of vector-valued or matrix-valued functions.

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§1. Statement of the problem. Results

1.1. The class of operators. Let $\Gamma \subset \mathbb{R}^d$ be a lattice, and let $\Omega \subset \mathbb{R}^d$ be the elementary cell of the lattice Γ. We denote $|\Omega| = \operatorname{meas}\Omega$. Below $\widetilde{H}^1(\Omega)$ stands for the subspace of functions in $H^1(\Omega)$ whose Γ-periodic extension to $\mathbb{R}^d$ belongs to $H^1_{\mathrm{loc}}(\mathbb{R}^d)$. If $\varphi(\mathbf{x})$ is a Γ-periodic function in $\mathbb{R}^d$, we denote

$$\varphi^\varepsilon(\mathbf{x}) := \varphi(\varepsilon^{-1}\mathbf{x}), \quad \varepsilon > 0.$$

Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain of class C^2. In $L_2(\mathcal{O};\mathbb{C}^n)$, we define an operator $\mathcal{A}_{D,\varepsilon}$ formally given by the differential expression

$$\mathcal{A}_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}) \tag{1.1}$$

with the Dirichlet condition on $\partial\mathcal{O}$. Here $g(\mathbf{x})$ is a measurable $(m\times m)$-matrix-valued function (in general, with complex entries). We assume that $g(\mathbf{x})$ is periodic with respect to the lattice Γ, bounded and uniformly positive definite. Next, $b(\mathbf{D}) = \sum_{l=1}^d b_l D_l$ is an $(m\times n)$-matrix first order DO with constant coefficients. Here b_l are constant matrices (in general, with complex entries). The symbol $b(\boldsymbol{\xi}) = \sum_{l=1}^d b_l \xi_l$, $\boldsymbol{\xi} \in \mathbb{R}^d$, corresponds to the operator $b(\mathbf{D})$. It is assumed that $m \ge n$ and that $\operatorname{rank} b(\boldsymbol{\xi}) = n$, $\forall \boldsymbol{\xi} \ne 0$. This condition is equivalent to the following inequalities

$$\alpha_0 \mathbf{1}_n \le b(\boldsymbol{\theta})^* b(\boldsymbol{\theta}) \le \alpha_1 \mathbf{1}_n, \quad \boldsymbol{\theta} \in \mathbb{S}^{d-1}, \quad 0 < \alpha_0 \le \alpha_1 < \infty, \tag{1.2}$$

with some positive constants α_0 and α_1.

The precise definition is the following: $\mathcal{A}_{D,\varepsilon}$ is the selfadjoint operator in $L_2(\mathcal{O};\mathbb{C}^n)$ generated by the quadratic form

$$a_{D,\varepsilon}[\mathbf{u},\mathbf{u}] = \int_{\mathcal{O}} \langle g^\varepsilon(\mathbf{x}) b(\mathbf{D})\mathbf{u}, b(\mathbf{D})\mathbf{u}\rangle \, d\mathbf{x}, \quad \mathbf{u} \in H^1_0(\mathcal{O};\mathbb{C}^n).$$

Under the above assumptions this form is closed in $L_2(\mathcal{O};\mathbb{C}^n)$ and positive definite. Moreover, we have

$$c_0 \int_{\mathcal{O}} |\mathbf{D}\mathbf{u}|^2 \, d\mathbf{x} \le a_{D,\varepsilon}[\mathbf{u},\mathbf{u}] \le c_1 \int_{\mathcal{O}} |\mathbf{D}\mathbf{u}|^2 \, d\mathbf{x}, \quad \mathbf{u} \in H^1_0(\mathcal{O};\mathbb{C}^n), \tag{1.3}$$

where $c_0 = \alpha_0 \|g^{-1}\|^{-1}_{L_\infty}$, $c_1 = \alpha_1 \|g\|_{L_\infty}$. It is easy to check (1.3) extending $\mathbf{u}$ by zero to $\mathbb{R}^d \setminus \mathcal{O}$, using the Fourier transformation and taking (1.2) into account.

The simplest example of the operator (1.1) is the scalar elliptic operator $\mathcal{A}_\varepsilon = -\operatorname{div} g^\varepsilon(\mathbf{x})\nabla = \mathbf{D}^* g^\varepsilon(\mathbf{x})\mathbf{D}$. In this case we have $n=1$, $m=d$,

$b(\mathbf{D}) = \mathbf{D}$. Obviously, condition (1.2) is valid with $\alpha_0 = \alpha_1 = 1$. Another example is the operator of elasticity theory which can be written in the form (1.1) with $n = d$, $m = d(d+1)/2$. These and other examples are discussed in [BSu1] in detail.

Our goal is to find approximation for small ε for the operator $\mathcal{A}_{D,\varepsilon}^{-1}$ in the operator norm in $L_2(\mathcal{O};\mathbb{C}^n)$. In terms of solutions, we are interested in the behavior of the generalized solution $\mathbf{u}_\varepsilon \in H_0^1(\mathcal{O};\mathbb{C}^n)$ of the Dirichlet problem

$$b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D})\mathbf{u}_\varepsilon(\mathbf{x}) = \mathbf{F}(\mathbf{x}), \quad \mathbf{x}\in\mathcal{O}; \quad \mathbf{u}_\varepsilon|_{\partial\mathcal{O}} = 0, \tag{1.4}$$

where $\mathbf{F} \in L_2(\mathcal{O};\mathbb{C}^n)$. Then $\mathbf{u}_\varepsilon = \mathcal{A}_{D,\varepsilon}^{-1}\mathbf{F}$.

1.2. The effective operator. In order to formulate the results, we need to introduce the effective operator $\mathcal{A}_D^0$.

Let an $(n\times m)$-matrix-valued function $\Lambda(\mathbf{x})$ be the (weak) Γ-periodic solution of the problem

$$b(\mathbf{D})^* g(\mathbf{x}) \left(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m\right) = 0, \qquad \int_\Omega \Lambda(\mathbf{x})\, d\mathbf{x} = 0. \tag{1.5}$$

In other words, for the columns $\mathbf{v}_j(\mathbf{x})$, $j = 1,\dots,m$, of the matrix $\Lambda(\mathbf{x})$ the following is true: $\mathbf{v}_j \in \widetilde{H}^1(\Omega;\mathbb{C}^n)$, we have

$$\int_\Omega \langle g(\mathbf{x})(b(\mathbf{D})\mathbf{v}_j(\mathbf{x}) + \mathbf{e}_j), b(\mathbf{D})\boldsymbol{\eta}(\mathbf{x})\rangle\, d\mathbf{x} = 0, \quad \forall \boldsymbol{\eta}\in \widetilde{H}^1(\Omega;\mathbb{C}^n),$$

and $\int_\Omega \mathbf{v}_j(\mathbf{x})\, d\mathbf{x} = 0$. Here $\mathbf{e}_1,\dots,\mathbf{e}_m$ is the standard orthonormal basis in $\mathbb{C}^m$.

The so-called *effective matrix* g^0 of size $m\times m$ is defined as follows:

$$g^0 = |\Omega|^{-1} \int_\Omega g(\mathbf{x}) \left(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m\right)\, d\mathbf{x}. \tag{1.6}$$

It turns out that the matrix (1.6) is positive definite. The *effective operator* $\mathcal{A}_D^0$ for $\mathcal{A}_{D,\varepsilon}$ is given by the differential expression

$$\mathcal{A}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$$

with the Dirichlet condition on $\partial\mathcal{O}$. The domain of this operator is $H_0^1(\mathcal{O};\mathbb{C}^n)\cap H^2(\mathcal{O};\mathbb{C}^n)$ (see Subsection 2.2 below).

Consider the "homogenized" Dirichlet problem

$$b(\mathbf{D})^* g^0 b(\mathbf{D})\mathbf{u}_0(\mathbf{x}) = \mathbf{F}(\mathbf{x}), \quad \mathbf{x}\in\mathcal{O}; \quad \mathbf{u}_0|_{\partial\mathcal{O}} = 0. \tag{1.7}$$

Then $\mathbf{u}_0 = (\mathcal{A}_D^0)^{-1}\mathbf{F}$. As $\varepsilon\to 0$, the solution $\mathbf{u}_\varepsilon$ of the problem (1.4) converges in $L_2(\mathcal{O};\mathbb{C}^n)$ to $\mathbf{u}_0$; for operators of the form (1.1) this was proved in [PSu]. We wish to estimate $\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathcal{O})}$.

1.3. The main result. Denote

$$(\partial\mathcal{O})_\varepsilon = \{\mathbf{x}\in\mathbb{R}^d :\ \operatorname{dist}\{\mathbf{x},\partial\mathcal{O}\} < \varepsilon\}.$$

Now we formulate the main result.

Theorem 1.1. *Assume that $\mathcal{O} \subset \mathbb{R}^d$ is a bounded domain of class C^2. Let $g(\mathbf{x})$ and $b(\mathbf{D})$ satisfy the assumptions of Subsection* 1.1. *Let $\mathbf{u}_\varepsilon$ be the solution of the problem* (1.4), *and let $\mathbf{u}_0$ be the solution of the problem* (1.7) *with $\mathbf{F} \in L_2(\mathcal{O};\mathbb{C}^n)$. Let $\varepsilon_1 \in (0,1]$ be such that the set $(\partial\mathcal{O})_{\varepsilon_1}$ can be covered by a finite number of open sets admitting diffeomorphisms of class C^2 rectifying the boundary $\partial\mathcal{O}$. Let $2r_1 = \operatorname{diam}\Omega$, $\varepsilon_2 = \varepsilon_1(1+r_1)^{-1}$, and $\varepsilon_0 = \varepsilon_2/2$. Then for $0 < \varepsilon \le \varepsilon_0$ we have*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathcal{O};\mathbb{C}^n)} \le C_1 \varepsilon \|\mathbf{F}\|_{L_2(\mathcal{O};\mathbb{C}^n)}, \tag{1.8}$$

or, in operator terms,

$$\|\mathcal{A}_{D,\varepsilon}^{-1} - (\mathcal{A}_D^0)^{-1}\|_{L_2(\mathcal{O};\mathbb{C}^n)\to L_2(\mathcal{O};\mathbb{C}^n)} \le C_1\varepsilon.$$

The constant C_1 depends only on m, d, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the parameters of the lattice Γ, and the domain $\mathcal{O}$.

§2. Auxiliary statements

2.1. The energy inequality. Consider the problem (1.4) with the right-hand side of class $H^{-1}(\mathcal{O};\mathbb{C}^n)$. Recall that $H^{-1}(\mathcal{O};\mathbb{C}^n)$ is defined as the space dual to $H_0^1(\mathcal{O};\mathbb{C}^n)$ with respect to the $L_2(\mathcal{O};\mathbb{C}^n)$-coupling. If $\mathbf{f} \in H^{-1}(\mathcal{O};\mathbb{C}^n)$ and $\boldsymbol{\eta} \in H_0^1(\mathcal{O};\mathbb{C}^n)$, then the symbol $(\mathbf{f},\boldsymbol{\eta})_{L_2(\mathcal{O})} = \int_{\mathcal{O}} \langle \mathbf{f},\boldsymbol{\eta}\rangle\, d\mathbf{x}$ stands for the value of the functional $\mathbf{f}$ on the element $\boldsymbol{\eta}$. Herewith,

$$\left|\int_{\mathcal{O}} \langle \mathbf{f},\boldsymbol{\eta}\rangle\, d\mathbf{x}\right| \le \|\mathbf{f}\|_{H^{-1}(\mathcal{O};\mathbb{C}^n)} \|\boldsymbol{\eta}\|_{H^1(\mathcal{O};\mathbb{C}^n)}.$$

The following (standard) statement was checked in [PSu, Lemma 4.1].

Lemma 2.1. *Let $\mathbf{f} \in H^{-1}(\mathcal{O};\mathbb{C}^n)$. Suppose that $\mathbf{z}_\varepsilon \in H_0^1(\mathcal{O};\mathbb{C}^n)$ is the generalized solution of the Dirichlet problem*

$$b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}) \mathbf{z}_\varepsilon(\mathbf{x}) = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathcal{O}; \quad \mathbf{z}_\varepsilon|_{\partial\mathcal{O}} = 0.$$

In other words, $\mathbf{z}_\varepsilon$ satisfies the identity

$$\int_{\mathcal{O}} \langle g^\varepsilon(\mathbf{x}) b(\mathbf{D})\mathbf{z}_\varepsilon, b(\mathbf{D})\boldsymbol{\eta}\rangle\, d\mathbf{x} = \int_{\mathcal{O}} \langle \mathbf{f},\boldsymbol{\eta}\rangle\, d\mathbf{x}, \quad \forall\, \boldsymbol{\eta} \in H_0^1(\mathcal{O};\mathbb{C}^n).$$

Then the following "energy inequality" is true:

$$\|\mathbf{z}_\varepsilon\|_{H^1(\mathcal{O};\mathbb{C}^n)} \le \widehat{C} \|\mathbf{f}\|_{H^{-1}(\mathcal{O};\mathbb{C}^n)},$$

where $\widehat{C} = (1 + (\operatorname{diam}\mathcal{O})^2)\alpha_0^{-1}\|g^{-1}\|_{L_\infty}$.

It follows from Lemma 2.1 that the operator $\mathcal{A}_{D,\varepsilon}^{-1}$ acting in $L_2(\mathcal{O};\mathbb{C}^n)$ can be extended to a linear continuous operator acting from $H^{-1}(\mathcal{O};\mathbb{C}^n)$ to $H_0^1(\mathcal{O};\mathbb{C}^n)$. Applying Lemma 2.1 with g^ε replaced by g^0, we see that the same statement is true for the operator $(\mathcal{A}_D^0)^{-1}$.

Note that

$$(\mathcal{A}_{D,\varepsilon}^{-1}\mathbf{f}_1, \mathbf{f}_2)_{L_2(\mathcal{O})} = (\mathbf{f}_1, \mathcal{A}_{D,\varepsilon}^{-1}\mathbf{f}_2)_{L_2(\mathcal{O})}, \quad \mathbf{f}_1, \mathbf{f}_2 \in H^{-1}(\mathcal{O};\mathbb{C}^n). \tag{2.1}$$

A similar identity is valid for the operator $(\mathcal{A}_D^0)^{-1}$.

All the statements of Subsection 2.1 are valid in arbitrary bounded domain $\mathcal{O}$ (without assumption that $\partial\mathcal{O} \in C^2$).

2.2. Properties of the solution of the homogenized problem. Due to the assumption $\partial\mathcal{O} \in C^2$, the solution $\mathbf{u}_0$ of the problem (1.7) satisfies $\mathbf{u}_0 \in H^1_0(\mathcal{O};\mathbb{C}^n) \cap H^2(\mathcal{O};\mathbb{C}^n)$, and

$$\|\mathbf{u}_0\|_{H^2(\mathcal{O};\mathbb{C}^n)} \leq \widehat{c}\|\mathbf{F}\|_{L_2(\mathcal{O};\mathbb{C}^n)}. \tag{2.2}$$

In operator terms, it means that

$$\|(\mathcal{A}^0_D)^{-1}\|_{L_2(\mathcal{O};\mathbb{C}^n)\to H^2(\mathcal{O};\mathbb{C}^n)} \leq \widehat{c}. \tag{2.3}$$

The constant $\widehat{c}$ depends only on α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and the domain $\mathcal{O}$. To justify these properties, it suffices to note that the operator $b(\mathbf{D})^*g^0b(\mathbf{D})$ is a *strongly elliptic* matrix DO and to apply the "additional smoothness" theorems for solutions of strongly elliptic systems (see, e. g., [McL, Chapter 4]).

2.3. Trace lemma. We need the following simple statement; see, e. g., [PSu, Lemma 5.1].

Lemma 2.2. *Denote $B_\varepsilon = \{\mathbf{x} \in \mathcal{O} : \operatorname{dist}\{\mathbf{x}, \partial\mathcal{O}\} < \varepsilon\}$. Then for any $z \in H^1(\mathcal{O})$ we have*

$$\int_{B_\varepsilon} |z|^2 d\mathbf{x} \leq \beta\varepsilon\|z\|_{H^1(\mathcal{O})}\|z\|_{L_2(\mathcal{O})}, \quad 0 < \varepsilon \leq \varepsilon_1.$$

Here ε_1 is the same as in Theorem 1.1. *The constant β depends only on the domain $\mathcal{O}$.*

Note that the statement of Lemma 2.2 is valid for any bounded domain $\mathcal{O}$ of class C^1.

2.4. Smoothing in Steklov's sense. Let S_ε be the operator in $L_2(\mathbb{R}^d;\mathbb{C}^m)$ given by

$$(S_\varepsilon\mathbf{u})(\mathbf{x}) = |\Omega|^{-1}\int_\Omega \mathbf{u}(\mathbf{x}-\varepsilon\mathbf{z})\,d\mathbf{z}. \tag{2.4}$$

It is said that the operator S_ε is *smoothing in Steklov's sense.*

We need the following property of the operator (2.4) (see [ZhPas, Lemma 1.1] or [PSu, Proposition 3.2]).

Lemma 2.3. *Let $f(\mathbf{x})$ be a Γ-periodic function in $\mathbb{R}^d$ such that $f \in L_2(\Omega)$. Let $[f^\varepsilon]$ denote the operator of multiplication by the function $f^\varepsilon(\mathbf{x})$. Then the operator $[f^\varepsilon]S_\varepsilon$ is continuous in $L_2(\mathbb{R}^d;\mathbb{C}^m)$, and*

$$\|[f^\varepsilon]S_\varepsilon\|_{L_2(\mathbb{R}^d;\mathbb{C}^m)\to L_2(\mathbb{R}^d;\mathbb{C}^m)} \leq |\Omega|^{-1/2}\|f\|_{L_2(\Omega)}.$$

2.5. Properties of the matrix $\Lambda(\mathbf{x})$. Let $\widetilde{\Gamma}$ be the lattice dual to Γ. By $\widetilde{\Omega}$ we denote the central Brillouin zone of $\widetilde{\Gamma}$, i. e., $\widetilde{\Omega} = \{\mathbf{k} \in \mathbb{R}^d : \ |\mathbf{k}| < |\mathbf{k}-\mathbf{b}|,\ 0 \neq \mathbf{b} \in \widetilde{\Gamma}\}$. Let r_0 be the radius of the ball inscribed in $\operatorname{clos}\widetilde{\Omega}$.

Recall that the matrix-valued function $\Lambda(\mathbf{x})$ is the Γ-periodic solution of the problem (1.5). In [BSu2, Subsection 7.3] it was proved that

$$\|\Lambda\|_{L_2(\Omega)} \leq m^{1/2}(2r_0)^{-1}|\Omega|^{1/2}\alpha_0^{-1/2}\|g\|_{L_\infty}^{1/2}\|g^{-1}\|_{L_\infty}^{1/2}. \tag{2.5}$$

Let $[\Lambda^\varepsilon]$ be the operator of multiplication by the matrix-valued function $\Lambda^\varepsilon(\mathbf{x})$; this operator acts from $L_2(\mathbb{R}^d;\mathbb{C}^m)$ to $L_2(\mathbb{R}^d;\mathbb{C}^n)$. By Lemma 2.3 and estimate (2.5), the norm of the operator $[\Lambda^\varepsilon]S_\varepsilon$ satisfies the following estimate:

$$\begin{aligned}\|[\Lambda^\varepsilon]S_\varepsilon\|_{L_2(\mathbb{R}^d;\mathbb{C}^m)\to L_2(\mathbb{R}^d;\mathbb{C}^n)} &\leq |\Omega|^{-1/2}\|\Lambda\|_{L_2(\Omega)}\\ &\leq m^{1/2}(2r_0)^{-1}\alpha_0^{-1/2}\|g\|_{L_\infty}^{1/2}\|g^{-1}\|_{L_\infty}^{1/2} =: M.\end{aligned} \tag{2.6}$$

§3. Proof of Theorem 1.1

The proof of Theorem 1.1 relies on the results of [PSu], where approximation of $\mathcal{A}_{D,\varepsilon}^{-1}$ in the norm of operators acting from $L_2(\mathcal{O};\mathbb{C}^n)$ to $H^1(\mathcal{O};\mathbb{C}^n)$ was obtained.

3.1. Error estimates in H^1. We fix a linear continuous extension operator

$$P_\mathcal{O} : H^2(\mathcal{O};\mathbb{C}^n) \to H^2(\mathbb{R}^d;\mathbb{C}^n) \tag{3.1}$$

and put $\widetilde{\mathbf{u}}_0 = P_\mathcal{O}\mathbf{u}_0$. Then

$$\|\widetilde{\mathbf{u}}_0\|_{H^2(\mathbb{R}^d;\mathbb{C}^n)} \leq C_\mathcal{O}\|\mathbf{u}_0\|_{H^2(\mathcal{O};\mathbb{C}^n)}, \tag{3.2}$$

where $C_\mathcal{O}$ is the norm of the operator (3.1). Let S_ε be the smoothing operator (2.4). By $R_\mathcal{O}$ we denote the operator of restriction of functions in $\mathbb{R}^d$ onto the domain $\mathcal{O}$. We put

$$K_D(\varepsilon) = R_\mathcal{O}[\Lambda^\varepsilon]S_\varepsilon b(\mathbf{D})P_\mathcal{O}(\mathcal{A}_D^0)^{-1}. \tag{3.3}$$

The operator $b(\mathbf{D})P_\mathcal{O}(\mathcal{A}_D^0)^{-1}$ is a continuous mapping of $L_2(\mathcal{O};\mathbb{C}^n)$ into $H^1(\mathbb{R}^d;\mathbb{C}^m)$. Using Lemma 2.3 and relation $\Lambda \in \widetilde{H}^1(\Omega)$, it is easy to check that the operator $[\Lambda^\varepsilon]S_\varepsilon$ is continuous from $H^1(\mathbb{R}^d;\mathbb{C}^m)$ to $H^1(\mathbb{R}^d;\mathbb{C}^n)$. Hence, the operator (3.3) is continuous from $L_2(\mathcal{O};\mathbb{C}^n)$ to $H^1(\mathcal{O};\mathbb{C}^n)$.

The following statement was proved in [PSu, (7.10)].

Proposition 3.1. *Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain of class C^2. Let $\mathbf{u}_\varepsilon$ be the solution of the problem* (1.4), *and let $\mathbf{u}_0$ be the solution of the problem* (1.7) *with $\mathbf{F} \in L_2(\mathcal{O};\mathbb{C}^n)$. Let $\widetilde{\mathbf{u}}_0 = P_\mathcal{O}\mathbf{u}_0$, where $P_\mathcal{O}$ is the extension operator* (3.1). *Let $\mathbf{w}_\varepsilon \in H^1(\mathcal{O};\mathbb{C}^n)$ be the generalized solution of the problem*

$$\mathcal{A}_\varepsilon\mathbf{w}_\varepsilon = 0 \text{ in } \mathcal{O}, \quad \mathbf{w}_\varepsilon|_{\partial\mathcal{O}} = \varepsilon\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0|_{\partial\mathcal{O}}. \tag{3.4}$$

Then for $0 < \varepsilon \leq 1$ we have

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0 + \mathbf{w}_\varepsilon\|_{H^1(\mathcal{O};\mathbb{C}^n)} \leq \widetilde{C}\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O};\mathbb{C}^n)}. \tag{3.5}$$

The constant $\widetilde{C}$ depends only on m, d, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the parameters of the lattice Γ and the domain $\mathcal{O}$.

The following theorem was proved in [PSu, Theorem 7.1].

Theorem 3.2. *Suppose that the assumptions of Theorem* 1.1 *are satisfied. Let* $\widetilde{\mathbf{u}}_0 = P_{\mathcal{O}}\mathbf{u}_0$, *where* $P_{\mathcal{O}}$ *is the extension operator* (3.1). *Then for* $0 < \varepsilon \le \varepsilon_2$ *we have*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{H^1(\mathcal{O};\mathbb{C}^n)} \le C\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O};\mathbb{C}^n)}, \tag{3.6}$$

or, in operator terms,

$$\|\mathcal{A}_{D,\varepsilon}^{-1} - (\mathcal{A}_D^0)^{-1} - \varepsilon K_D(\varepsilon)\|_{L_2(\mathcal{O};\mathbb{C}^n)\to H^1(\mathcal{O};\mathbb{C}^n)} \le C\varepsilon^{1/2}.$$

The constant C *depends only on* m, d, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *the parameters of the lattice* Γ, *and the domain* $\mathcal{O}$.

Recall that $(\partial\mathcal{O})_\varepsilon$ denotes the ε-neighborhood of $\partial\mathcal{O}$. For sufficiently small ε, we fix two cut-off functions $\theta_\varepsilon(\mathbf{x})$ and $\widetilde{\theta}_\varepsilon(\mathbf{x})$ in $\mathbb{R}^d$ such that

$$\begin{aligned} &\theta_\varepsilon \in C_0^\infty(\mathbb{R}^d), \quad \operatorname{supp}\theta_\varepsilon \subset (\partial\mathcal{O})_\varepsilon, \quad 0 \le \theta_\varepsilon(\mathbf{x}) \le 1, \\ &\theta_\varepsilon(\mathbf{x})|_{\partial\mathcal{O}} = 1, \quad \varepsilon\,|\nabla\theta_\varepsilon(\mathbf{x})| \le \kappa = \text{const}; \end{aligned} \tag{3.7}$$

$$\begin{aligned} &\widetilde{\theta}_\varepsilon \in C_0^\infty(\mathbb{R}^d), \quad \operatorname{supp}\widetilde{\theta}_\varepsilon \subset (\partial\mathcal{O})_{2\varepsilon}, \quad 0 \le \widetilde{\theta}_\varepsilon(\mathbf{x}) \le 1, \\ &\widetilde{\theta}_\varepsilon(\mathbf{x}) = 1 \text{ for } \mathbf{x} \in (\partial\mathcal{O})_\varepsilon, \quad \varepsilon\left|\nabla\widetilde{\theta}_\varepsilon(\mathbf{x})\right| \le \widetilde{\kappa} = \text{const}. \end{aligned} \tag{3.8}$$

We denote

$$\boldsymbol{\phi}_\varepsilon = \varepsilon\theta_\varepsilon\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0. \tag{3.9}$$

From (1.2), (2.2), (2.6), (3.2), and (3.7) it follows that

$$\|\boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O};\mathbb{C}^n)} \le \varepsilon M\alpha_1^{1/2}\|\widetilde{\mathbf{u}}_0\|_{H^1(\mathbb{R}^d;\mathbb{C}^n)} \le \varepsilon M\alpha_1^{1/2}C_{\mathcal{O}}\widehat{c}\|\mathbf{F}\|_{L_2(\mathcal{O};\mathbb{C}^n)}, \tag{3.10}$$

cf. [PSu, (7.14)]. The norm of the function (3.9) in $H^1(\mathcal{O};\mathbb{C}^n)$ was estimated in [PSu, Lemma 7.4]. A similar estimate is true if θ_ε is replaced by $\widetilde{\theta}_\varepsilon$. We formulate the corresponding result.

Lemma 3.3. *Suppose that the assumptions of Theorem* 1.1 *are satisfied. Let* θ_ε *and* $\widetilde{\theta}_\varepsilon$ *be functions satisfying* (3.7), (3.8), *and let* $\boldsymbol{\phi}_\varepsilon$ *be defined by* (3.9). *Then we have*

$$\|\boldsymbol{\phi}_\varepsilon\|_{H^1(\mathcal{O};\mathbb{C}^n)} \le C_2\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O};\mathbb{C}^n)}, \quad 0 < \varepsilon \le \varepsilon_2, \tag{3.11}$$

$$\|\varepsilon\widetilde{\theta}_\varepsilon\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{H^1(\mathcal{O};\mathbb{C}^n)} \le \widetilde{C}_2\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O};\mathbb{C}^n)}, \quad 0 < 2\varepsilon \le \varepsilon_2. \tag{3.12}$$

The constants C_2 *and* $\widetilde{C}_2$ *depend only on* m, d, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *the parameters of the lattice* Γ, *and the domain* $\mathcal{O}$.

3.2. Proof of Theorem 1.1. Step 1. Roughening (3.5), we obtain

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0 + \mathbf{w}_\varepsilon\|_{L_2(\mathcal{O};\mathbb{C}^n)} \le \widetilde{C}\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O};\mathbb{C}^n)}, \quad 0 < \varepsilon \le 1. \tag{3.13}$$

Combining (1.2), (2.2), (2.6), and (3.2), we see that

$$\|\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{L_2(\mathcal{O})} \le M\alpha_1^{1/2}C_{\mathcal{O}}\widehat{c}\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{3.14}$$

From (3.13) and (3.14) it follows that

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathcal{O})} \le \varepsilon(\widetilde{C} + M\alpha_1^{1/2} C_{\mathcal{O}}\widehat{c})\|\mathbf{F}\|_{L_2(\mathcal{O})} + \|\mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})}, \quad 0 < \varepsilon \le 1. \tag{3.15}$$

Therefore, the proof of estimate (1.8) is reduced to estimating of $\mathbf{w}_\varepsilon$ in $L_2(\mathcal{O};\mathbb{C}^n)$.

For this purpose, we need the following lemma.

Lemma 3.4. *Suppose that the assumptions of Theorem* 1.1 *are satisfied. Let $\widetilde{\theta}_\varepsilon$ be a function satisfying* (3.8). *Consider the operator*

$$\widetilde{K}_D(\varepsilon) = R_{\mathcal{O}}[(1-\widetilde{\theta}_\varepsilon)\Lambda^\varepsilon] S_\varepsilon b(\mathbf{D}) P_{\mathcal{O}} (\mathcal{A}_D^0)^{-1}, \tag{3.16}$$

which is a continuous mapping of $L_2(\mathcal{O};\mathbb{C}^n)$ into $H_0^1(\mathcal{O};\mathbb{C}^n)$. Let $(\widetilde{K}_D(\varepsilon))^* : H^{-1}(\mathcal{O};\mathbb{C}^n) \to L_2(\mathcal{O};\mathbb{C}^n)$ be the operator adjoint to the operator (3.16), *i. e.*,

$$\begin{gathered}\Big((\widetilde{K}_D(\varepsilon))^*\mathbf{f}, \mathbf{v}\Big)_{L_2(\mathcal{O})} = \Big(\mathbf{f}, \widetilde{K}_D(\varepsilon)\mathbf{v}\Big)_{L_2(\mathcal{O})}, \\ \forall\, \mathbf{f} \in H^{-1}(\mathcal{O};\mathbb{C}^n), \quad \forall\, \mathbf{v} \in L_2(\mathcal{O};\mathbb{C}^n).\end{gathered} \tag{3.17}$$

Then the operator $\mathcal{A}_{D,\varepsilon}^{-1}$, viewed as a continuous mapping of $H^{-1}(\mathcal{O};\mathbb{C}^n)$ into $L_2(\mathcal{O};\mathbb{C}^n)$, admits the following approximation

$$\|\mathcal{A}_{D,\varepsilon}^{-1} - (\mathcal{A}_D^0)^{-1} - \varepsilon(\widetilde{K}_D(\varepsilon))^*\|_{H^{-1}(\mathcal{O})\to L_2(\mathcal{O})} \le (C + \widetilde{C}_2)\varepsilon^{1/2}, \quad 0 < 2\varepsilon \le \varepsilon_2. \tag{3.18}$$

Proof. From (3.6) and (3.12) it follows that

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon(1-\widetilde{\theta}_\varepsilon)\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0\|_{H^1(\mathcal{O};\mathbb{C}^n)} \le (C + \widetilde{C}_2)\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O};\mathbb{C}^n)}, \quad 0 < 2\varepsilon \le \varepsilon_2. \tag{3.19}$$

The function under the norm-sign on the left belongs to $H_0^1(\mathcal{O};\mathbb{C}^n)$. In operator terms, (3.19) means that

$$\|\mathcal{A}_{D,\varepsilon}^{-1} - (\mathcal{A}_D^0)^{-1} - \varepsilon\widetilde{K}_D(\varepsilon)\|_{L_2(\mathcal{O})\to H_0^1(\mathcal{O})} \le (C + \widetilde{C}_2)\varepsilon^{1/2}, \quad 0 < 2\varepsilon \le \varepsilon_2. \tag{3.20}$$

This implies (3.18) by the duality arguments. Indeed, combining (2.1), the similar identity for $(\mathcal{A}_D^0)^{-1}$ and (3.17), we see that for any $\mathbf{f} \in H^{-1}(\mathcal{O};\mathbb{C}^n)$ and $\mathbf{v} \in L_2(\mathcal{O};\mathbb{C}^n)$ one has

$$\begin{aligned}&\Big((\mathcal{A}_{D,\varepsilon}^{-1} - (\mathcal{A}_D^0)^{-1} - \varepsilon(\widetilde{K}_D(\varepsilon))^*)\mathbf{f}, \mathbf{v}\Big)_{L_2(\mathcal{O})} \\ &= \Big(\mathbf{f}, (\mathcal{A}_{D,\varepsilon}^{-1} - (\mathcal{A}_D^0)^{-1} - \varepsilon\widetilde{K}_D(\varepsilon))\mathbf{v}\Big)_{L_2(\mathcal{O})}.\end{aligned}$$

Together with (3.20) this yields

$$\begin{aligned}&\left|\Big((\mathcal{A}_{D,\varepsilon}^{-1} - (\mathcal{A}_D^0)^{-1} - \varepsilon(\widetilde{K}_D(\varepsilon))^*)\mathbf{f}, \mathbf{v}\Big)_{L_2(\mathcal{O})}\right| \\ &\le (C + \widetilde{C}_2)\varepsilon^{1/2}\|\mathbf{f}\|_{H^{-1}(\mathcal{O})}\|\mathbf{v}\|_{L_2(\mathcal{O})}, \quad \forall\, \mathbf{f} \in H^{-1}(\mathcal{O};\mathbb{C}^n), \quad \forall\, \mathbf{v} \in L_2(\mathcal{O};\mathbb{C}^n),\end{aligned}$$

which implies (3.18). $\bullet$

From (3.7) and (3.9) it follows that $\boldsymbol{\phi}_\varepsilon|_{\partial\mathcal{O}} = \varepsilon\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0|_{\partial\mathcal{O}}$. Then, by (3.4), the function $\mathbf{w}_\varepsilon - \boldsymbol{\phi}_\varepsilon$ is the solution of the problem

$$\mathcal{A}_\varepsilon(\mathbf{w}_\varepsilon - \boldsymbol{\phi}_\varepsilon) = \mathbf{F}_\varepsilon \text{ in } \mathcal{O}, \quad (\mathbf{w}_\varepsilon - \boldsymbol{\phi}_\varepsilon)|_{\partial\mathcal{O}} = 0, \tag{3.21}$$

where $\mathbf{F}_\varepsilon = -\mathcal{A}_\varepsilon\boldsymbol{\phi}_\varepsilon$. It is easily seen that $\mathbf{F}_\varepsilon \in H^{-1}(\mathcal{O};\mathbb{C}^n)$, and

$$\|\mathbf{F}_\varepsilon\|_{H^{-1}(\mathcal{O};\mathbb{C}^n)} \le \alpha_1 d^{1/2}\|g\|_{L_\infty}\|\boldsymbol{\phi}_\varepsilon\|_{H^1(\mathcal{O};\mathbb{C}^n)}, \tag{3.22}$$

see [PSu, (4.15)]. From (3.11) and (3.22) it follows that

$$\|\mathbf{F}_\varepsilon\|_{H^{-1}(\mathcal{O};\mathbb{C}^n)} \le C_3\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(\mathcal{O};\mathbb{C}^n)}, \quad 0<\varepsilon\le\varepsilon_2, \tag{3.23}$$

where $C_3 = \alpha_1 d^{1/2}\|g\|_{L_\infty}C_2$. Note also that $\mathbf{F}_\varepsilon$ is supported in $(\partial\mathcal{O})_\varepsilon$.

Now we apply approximation (3.18) to the problem (3.21). Since $\mathbf{w}_\varepsilon - \boldsymbol{\phi}_\varepsilon = \mathcal{A}_{D,\varepsilon}^{-1}\mathbf{F}_\varepsilon$, then

$$\|\mathbf{w}_\varepsilon - \boldsymbol{\phi}_\varepsilon - (\mathcal{A}_D^0)^{-1}\mathbf{F}_\varepsilon - \varepsilon(\widetilde{K}_D(\varepsilon))^*\mathbf{F}_\varepsilon\|_{L_2(\mathcal{O})} \le (C+\widetilde{C}_2)\varepsilon^{1/2}\|\mathbf{F}_\varepsilon\|_{H^{-1}(\mathcal{O})}, \quad 0<2\varepsilon\le\varepsilon_2. \tag{3.24}$$

By (3.16) and (3.17), for any $\mathbf{v}\in L_2(\mathcal{O};\mathbb{C}^n)$ we have

$$\Big((\widetilde{K}_D(\varepsilon))^*\mathbf{F}_\varepsilon,\mathbf{v}\Big)_{L_2(\mathcal{O})} = \Big(\mathbf{F}_\varepsilon,(1-\widetilde{\theta}_\varepsilon)\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})P_{\mathcal{O}}(\mathcal{A}_D^0)^{-1}\mathbf{v}\Big)_{L_2(\mathcal{O})}. \tag{3.25}$$

Since $1-\widetilde{\theta}_\varepsilon(\mathbf{x}) = 0$ for $\operatorname{dist}\{\mathbf{x},\partial\mathcal{O}\}\le\varepsilon$, and $\mathbf{F}_\varepsilon$ is supported in $(\partial\mathcal{O})_\varepsilon$, then the right-hand side of (3.25) is equal to zero. Consequently, $(\widetilde{K}_D(\varepsilon))^*\mathbf{F}_\varepsilon = 0$.

Then (3.24) and (3.23) imply that

$$\|\mathbf{w}_\varepsilon - \boldsymbol{\phi}_\varepsilon - (\mathcal{A}_D^0)^{-1}\mathbf{F}_\varepsilon\|_{L_2(\mathcal{O})} \le (C+\widetilde{C}_2)C_3\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}, \quad 0<2\varepsilon\le\varepsilon_2. \tag{3.26}$$

The norm of $\boldsymbol{\phi}_\varepsilon$ in $L_2(\mathcal{O};\mathbb{C}^n)$ admits estimate (3.10). It remains to estimate the L_2-norm of the function $(\mathcal{A}_D^0)^{-1}\mathbf{F}_\varepsilon$.

3.3. Proof of Theorem 1.1. Step 2.

Lemma 3.5. *Suppose that the assumptions of Theorem* 1.1 *are satisfied. Let $\boldsymbol{\phi}_\varepsilon$ be defined by* (3.9)*, and let $\mathbf{F}_\varepsilon = -\mathcal{A}_\varepsilon\boldsymbol{\phi}_\varepsilon = -b(\mathbf{D})^*g^\varepsilon b(\mathbf{D})\boldsymbol{\phi}_\varepsilon$. Then the function $\boldsymbol{\eta}_\varepsilon := (\mathcal{A}_D^0)^{-1}\mathbf{F}_\varepsilon$ satisfies the following estimate*:

$$\|\boldsymbol{\eta}_\varepsilon\|_{L_2(\mathcal{O};\mathbb{C}^n)} \le C_4\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O};\mathbb{C}^n)}, \quad 0<\varepsilon\le\varepsilon_2. \tag{3.27}$$

The constant C_4 depends only on m, d, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the parameters of the lattice Γ, and the domain $\mathcal{O}$.

Proof. The function $\boldsymbol{\eta}_\varepsilon \in H_0^1(\mathcal{O};\mathbb{C}^n)$ is the generalized solution of the Dirichlet problem $\mathcal{A}^0\boldsymbol{\eta}_\varepsilon = \mathbf{F}_\varepsilon$, $\boldsymbol{\eta}_\varepsilon|_{\partial\mathcal{O}} = 0$. It means that

$$\int_{\mathcal{O}}\langle g^0 b(\mathbf{D})\boldsymbol{\eta}_\varepsilon, b(\mathbf{D})\mathbf{h}\rangle\,d\mathbf{x} = \int_{\mathcal{O}}\langle\mathbf{F}_\varepsilon,\mathbf{h}\rangle\,d\mathbf{x} = -\int_{\mathcal{O}}\langle g^\varepsilon b(\mathbf{D})\boldsymbol{\phi}_\varepsilon, b(\mathbf{D})\mathbf{h}\rangle\,d\mathbf{x}, \quad \forall\,\mathbf{h}\in H_0^1(\mathcal{O};\mathbb{C}^n). \tag{3.28}$$

If $\mathbf{h} \in H^2(\mathcal{O};\mathbb{C}^n) \cap H^1_0(\mathcal{O};\mathbb{C}^n)$, then it is possible to integrate by parts in the left-hand side of (3.28). Hence,

$$\begin{aligned} \int_{\mathcal{O}} \langle \boldsymbol{\eta}_\varepsilon, \mathcal{A}^0 \mathbf{h} \rangle \, d\mathbf{x} = - \int_{\mathcal{O}} \langle g^\varepsilon b(\mathbf{D}) \boldsymbol{\phi}_\varepsilon, b(\mathbf{D})\mathbf{h} \rangle \, d\mathbf{x}, \\ \forall\, \mathbf{h} \in H^2(\mathcal{O};\mathbb{C}^n) \cap H^1_0(\mathcal{O};\mathbb{C}^n). \end{aligned} \tag{3.29}$$

Now we write down the norm of the function $\boldsymbol{\eta}_\varepsilon$ in $L_2(\mathcal{O};\mathbb{C}^n)$ as the norm of continuous antilinear functional:

$$\|\boldsymbol{\eta}_\varepsilon\|_{L_2(\mathcal{O})} = \sup_{0 \neq \mathbf{G} \in L_2(\mathcal{O};\mathbb{C}^n)} \frac{\left| \int_{\mathcal{O}} \langle \boldsymbol{\eta}_\varepsilon, \mathbf{G} \rangle \, d\mathbf{x} \right|}{\|\mathbf{G}\|_{L_2(\mathcal{O})}}.$$

We put $\mathbf{h} = (\mathcal{A}^0_D)^{-1}\mathbf{G}$, $\mathbf{G} \in L_2(\mathcal{O};\mathbb{C}^n)$. Then $\mathbf{G} = \mathcal{A}^0\mathbf{h}$, and $\mathbf{h}$ runs through $H^2(\mathcal{O};\mathbb{C}^n) \cap H^1_0(\mathcal{O};\mathbb{C}^n)$ if $\mathbf{G}$ runs through $L_2(\mathcal{O};\mathbb{C}^n)$ (see Subsection 2.2). Hence,

$$\|\boldsymbol{\eta}_\varepsilon\|_{L_2(\mathcal{O})} = \sup_{0 \neq \mathbf{h} \in H^2(\mathcal{O}) \cap H^1_0(\mathcal{O})} \frac{\left| \int_{\mathcal{O}} \langle \boldsymbol{\eta}_\varepsilon, \mathcal{A}^0 \mathbf{h} \rangle \, d\mathbf{x} \right|}{\|\mathcal{A}^0\mathbf{h}\|_{L_2(\mathcal{O})}}. \tag{3.30}$$

By (2.3), we have $\|\mathcal{A}^0\mathbf{h}\|_{L_2(\mathcal{O})} \geq (\widehat{c})^{-1}\|\mathbf{h}\|_{H^2(\mathcal{O})}$. Combining this with (3.29) and (3.30), we obtain

$$\|\boldsymbol{\eta}_\varepsilon\|_{L_2(\mathcal{O})} \leq \widehat{c} \sup_{0 \neq \mathbf{h} \in H^2(\mathcal{O}) \cap H^1_0(\mathcal{O})} \frac{\left| \int_{\mathcal{O}} \langle g^\varepsilon b(\mathbf{D}) \boldsymbol{\phi}_\varepsilon, b(\mathbf{D})\mathbf{h} \rangle \, d\mathbf{x} \right|}{\|\mathbf{h}\|_{H^2(\mathcal{O})}}. \tag{3.31}$$

Next, since $b(\mathbf{D}) = \sum_{l=1}^d b_l D_l$ and, by (1.2), $|b_l| \leq \alpha_1^{1/2}$, then

$$\|g^\varepsilon b(\mathbf{D}) \boldsymbol{\phi}_\varepsilon\|_{L_2(\mathcal{O})} \leq \|g\|_{L_\infty} \alpha_1^{1/2} d^{1/2} \|\boldsymbol{\phi}_\varepsilon\|_{H^1(\mathcal{O})}. \tag{3.32}$$

Taking into account that the function $\boldsymbol{\phi}_\varepsilon$ is supported in the ε-neighborhood of $\partial\mathcal{O}$, from (3.31) and (3.32) we see that

$$\|\boldsymbol{\eta}_\varepsilon\|_{L_2(\mathcal{O})} \leq \widehat{c}\|g\|_{L_\infty} \alpha_1^{1/2} d^{1/2} \|\boldsymbol{\phi}_\varepsilon\|_{H^1(\mathcal{O})} \sup_{0 \neq \mathbf{h} \in H^2(\mathcal{O}) \cap H^1_0(\mathcal{O})} \frac{\|b(\mathbf{D})\mathbf{h}\|_{L_2(B_\varepsilon)}}{\|\mathbf{h}\|_{H^2(\mathcal{O})}}. \tag{3.33}$$

Applying Lemma 2.2 and taking into account that $|b(\mathbf{D})\mathbf{h}| \leq \alpha_1^{1/2} \sum_{l=1}^d |D_l \mathbf{h}|$, for $0 < \varepsilon \leq \varepsilon_1$ we have:

$$\begin{aligned} \int_{B_\varepsilon} |b(\mathbf{D})\mathbf{h}|^2 \, d\mathbf{x} &\leq \alpha_1 d \sum_{l=1}^d \int_{B_\varepsilon} |D_l \mathbf{h}|^2 \, d\mathbf{x} \\ &\leq \alpha_1 d \beta \varepsilon \sum_{l=1}^d \|D_l \mathbf{h}\|_{H^1(\mathcal{O})} \|D_l \mathbf{h}\|_{L_2(\mathcal{O})} \leq \alpha_1 d \beta \varepsilon \|\mathbf{h}\|_{H^2(\mathcal{O})} \|\mathbf{h}\|_{H^1(\mathcal{O})}. \end{aligned}$$

Hence,

$$\sup_{0 \neq \mathbf{h} \in H^2(\mathcal{O}) \cap H^1_0(\mathcal{O})} \frac{\|b(\mathbf{D})\mathbf{h}\|_{L_2(B_\varepsilon)}}{\|\mathbf{h}\|_{H^2(\mathcal{O})}} \leq (\alpha_1 d \beta)^{1/2} \varepsilon^{1/2}, \quad 0 < \varepsilon \leq \varepsilon_1. \tag{3.34}$$

Finally, from (3.11), (3.33), and (3.34) it follows that (3.27) is valid with $C_4 = \widehat{c}\|g\|_{L_\infty}\beta^{1/2}\alpha_1 d C_2$. $\bullet$

Now it is easy to complete the **proof of Theorem 1.1**. By (3.10), (3.26), and (3.27), we have

$$\|\mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})} \le C_5\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}, \quad 0 < 2\varepsilon \le \varepsilon_2,$$

where $C_5 = (C + \widetilde{C}_2)C_3 + M\alpha_1^{1/2}C_{\mathcal{O}}\widehat{c} + C_4$. Combining this with (3.15), we arrive at (1.8) with $C_1 = \widetilde{C} + M\alpha_1^{1/2}C_{\mathcal{O}}\widehat{c} + C_5$. $\bullet$

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